Scalar field theory on κ-Minkowski spacetime and translation and Lorentz invariance

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We investigate the properties of κ-Minkowski spacetime by using representations of the corresponding deformed algebra in terms of undeformed Heisenberg-Weyl algebra. The deformed algebra consists of κ-Poincaré algebra extended with the generators of the deformed Weyl algebra. The part of deformed algebra, generated by rotation, boost and momentum generators, is described by the Hopf algebra structure. The approach used in our considerations is completely Lorentz covariant. We further use an advantages of this approach to consistently construct a star product which has a property that under integration sign it can be replaced by a standard pointwise multiplication, a property that was since known to hold for Moyal, but not also for κ-Minkowski spacetime. This star product also has generalized trace and cyclic properties and the construction alone is accomplished by considering a classical Dirac operator representation of deformed algebra and by requiring it to be hermitian. We find that the obtained star product is not translationally invariant, leading to a conclusion that the classical Dirac operator representation is the one where translation invariance cannot simultaneously be implemented along with hermiticity. However, due to the integral property satisfied by the star product, noncommutative free scalar field theory does not have a problem with translation symmetry breaking and can be shown to reduce to an ordinary free scalar field theory without nonlocal features and tachionic modes and basicaly of the very same form. The issue of Lorentz invariance of the theory is also discussed.

1. INTRODUCTION

There are two basic areas of interest which make κ-Minkowski spacetime especially interesting object of theoretical investigation from both, physical as well as mathematical perspective. Since it emerges naturally from κ-Poincaré algebra [1],[2], which provides a group theoretical framework for describing symmetry lying in the core of the Doubly Special Relativity (DSR) theories [3],[4],[5],[6], it is thus convenient spacetime candidate for DSR theories. Although different proposals for DSR theories can be looked upon as different bases [7],[8] for κ-Poincaré algebra, they all have in their core the very same noncommutative structure, encoded within κ-deformed algebra. Other important argument that favours κ-Minkowski spacetime is the strong indication that it as well arises in the context of quantum gravity coupled to matter fields [9],[10]. These considerations show that after integrating out topological degrees of freedom of gravity, the effective dynamics of matter fields is described by a noncommutative quantum field theory which has a κ-Poincaré group as its symmetry group [11],[12],[13]. In this context the κ-Minkowski can be, and there are some arguments to support this, considered as a flat limit of quantum gravity in pretty much the same way as the special relativity corresponds to general relativity when the same limit is concerned.

This situation makes a noncommutative field theories on noncommutative spacetimes, particularly those of the κ-Minkowski type, even more interesting subject to study. Various attempts have been undertaken towards this direction by many authors [14],[15],[16],[17],[18], including various possibilities for constructing field theories on κ-Minkowski spacetime and investigating their properties. Recently it is well established that if κ-Poincaré Hopf algebra is supposed to be a plausible model for describing physics in κ-Minkowski spacetime, then it is necessary to accept certain modifications in statistics obeyed by the particles. This means that κ-Minkowski spacetime leads to modification of particle

statistics which results in deformed oscillator algebras [19],[20],[21],[22],[23],[24],[25]. Deformation quantization of Poincaré algebra can be performed by means of the twist operator [26],[27],[28] which happens to include dilatation generator, thus belonging to the universal enveloping algebra of the general linear algebra [29],[30],[31],[32],[33],[34]. This twist operator gives rise to a deformed statistics on κ-Minkowski spacetime [29],[24],[35],[36]. In correspondence to these observations, bounds can be put on the quantum gravity scale by using deformed statistics results, in the context of atomic physics [37],[38] as well as by comparing deformed dispersion relations to corresponding time delay calculations of high energy photons [39].

During last decade or so there has been quite large effort made to formulate integration on κ-Minkowski and to find a relevant meassure that would consequently be used in formulating field theory. The issue of proper definition of the integral on κ-Minkowski spacetime, that would have desired trace and cyclic properties, was considered by various groups of authors. In Refs.[40],[18],[41],[42],[43] the problem of cyclicity of the integral is tackled in a way that required the introduction of a new integration meassure which then had to satisfy certain conditions, stated in a form of differential equations. On the other hand, in Ref.[44] the authors were considering a star product for noncommutative spaces of Lie type, which provides a definition of an invariant integral, satisfying quasicyclicity property. According to their claims, this quasicyclicity property reduces to exact cyclicity in the case when the adjoint representation of the underlying Lie algebra is traceless. Several other attempts for identifying a proper integral identity that would enclose most of peculiarities of κ-Minkowski have been made. In Ref.[45] an example of such integral identity is obtained which has an adventage that it does not require a modification of an integration meassure, but deals with a star product that leads to unphysical modes, such as the appeareance of tachyons. The appeareance of tachyonic modes draws its origin in basicaly nonlocal character of the action obtained in [45]. The same identity is again mentioned in [46], and a similar one is redireved in Ref.[47] in a more general setting corresponding to a general class of various orderings.

In the present paper we are further investigating the basic mathematical properties of κ-Minkowski spacetime and consequences they have on the physics, especially on the matters regarding nonlocality and translation and Lorentz invariance. Particularly we shall show that it is possible to introduce a star product in κ-Minkowski spacetime, that, under the integration sign, can be replaced by the standard pointwise multiplication. The procedure for showing this is governed by simple physical principles and appears to be internally consistent. The same star product also happens to have a generalized trace and cyclic properties that can reduce to standard trace and cyclic properties, if certain conditions, imposed on physical fields, are satisfied. One additional point that is specific about our approach is that the issue with meassure on κ-Minkowski is in a sort of way avoided by absorbing it within a new star product in such a way that the new star product has a correct limit when the parameter of deformation goes to zero. All this can be shown within our approach that uses a method of realizations. In this method the generators of deformed algebra, which include noncommutative coordinates and generators of the Poincaré algebra, are represented in terms of generators of the undeformed Heisenberg algebra, i.e. in terms of ordinary coordinates and their respective derivatives. The important point about this procedure is that from the beginning, we start with the integral on κ-Minkowski having ordinary integration meassure and demand that realization of deformed algebra be hermitian. The stated requirement shows up as a crucial step, which allows for the construction of the star product, whose generalized trace and cyclic properties then emerge as a beneficial side effects. In this way the generalized trace property of the integral on κ-Minkowski spacetime arises naturally, simply by requiring from the realization of deformed algebra to be hermitian. Though, the star product resulting from the hermitization procedure appears to be translationally noninvariant. This feature shows that in the classical Dirac operator representation, the one in which we are making our analysis, hermiticity and translation invariance mutually interfere in a sense that it is not possible to simultaneously have both of these properties satisfied on κ-Minkowski, at least as far as the classical Dirac operator representation is concerned.

The symmetry underlying κ-Minkowski spacetime is described by the κ-deformed Poincaré algebra, whose mathematical structure is most conveniently specified with Hopf algebra. The investigations here will be pursued in a so called classical basis [7],[8],[47],[49],[50] where the algebraic part of κ-Poincaré Hopf algebra is undeformed and all deformations are contained within the coalgebraic sector. This means that the action of Poincaré generators on κ-Minkowski spacetime and consequently on the algebra of noncommutative functions will change with respect to their

action on ordinary Minkowski space, giving rise to the appropriate modifications of Leibniz rules and corresponding coproducts. The approach followed in this paper is developed and widely elaborated in a series of papers [47],[50],[51], [52],[53],[54],[55] that are concerned with classifying of noncommutative spaces and investigations of various properties of their realizations. The general principle established there is that to each realization of noncommutative space, there corresponds a particular ordering prescription and definite coproduct and star product as well as a twist operator. Here we shall use one particular realization, the so called classical Dirac operator representation, that in this series of papers is referred to as the *natural realization* and which is characterized by the simple requirement that the deformed and undeformed derivatives have to be the same, i.e. identified. The classical Dirac operator representation was explicitly or implicitly considered by many authors [56],[57],[18],[41],[42],[45] and it seems to be of special importance to physics since it can be related to a generalized uncertainty principle that has its origin in the study of high energy collisions of strings [58]. It is thus also directly related to the existence of a minimal length [59],[60]. For this classical Dirac operator representation we find a corresponding star product. After requiring this realization to be hermitian, we arrive at the result that, under the integration sign, the corresponding star product can be replaced with the pointwise multiplication and also happens to satisfy a generalized trace property. The most of methods used in investigations that were performed here are taken over from the Fock space analysis carried out in [61],[62],[63],[64],[65],[66],[67].

The plan of paper is the following. In section 2 we introduce an algebraic setting we are working in. It includes the algebra of coordinates on κ-Minkowski spacetime, an algebra of symmetry generators, which we take to be a undeformed Poincaré algebra, and the action of symmetry generators on κ-Minkowski spacetime. It also includes notion of a shift operator and a definition of the star product that we shall use throughout this paper. In this section we also introduce a classical Dirac operator representation of the κ-deformed algebra and the star product that we use is defined with respect to this representation. In section 3 we further elaborate on the algebraic properties of κ-Minkowski spacetime and find that these properties are enclosed within a Hopf algebra structure. In particular we list the coproducts and antipodes of κ-Poincaré symmetry generators. Section 4 is devoted to analyzing the properties of the star product corresponding to classical Dirac operator representation (*natural realization*) and an integral identity satisfied by the star product is there rederived in a new way, exposing its nonlocal properties. This star product, before hermitization is carried out, is shown to be translationally invariant. The process of hermitization of the classical Dirac operator representation is then carried out in section 5 and is found to lead to a cancellation of nonlocal operators appearing in the theory, giving rise to generalized trace and cyclic properties of the star product. The hermitization procedure also enables a proper and consistent introduction of hermitian conjugation (i.e. adjoint operation) and makes a correspondence between commutative and noncommutative algebras of functions fully established. However, the star product that is obtained in this way, as a result of hermitization, is not translationally invariant. In the conclusion we further discuss this isssue of translational noninvariance of the hermitized star product and finish the paper with discussion regarding the problems with Lorentz invariance of the theory and conclude by suggesting a proposal about a way in which Lorentz symmetry can be restored.

2. κ-MINKOWSKI SPACETIME

We consider a κ-deformed Minkowski spacetime whose noncommutative coordinates $\hat{x}_\mu$, $(\mu = 0, 1, ..., n-1)$, close a Lie algebra with the Lorentz generators $M_{\mu\nu}$, $(M_{\mu\nu} = -M_{\nu\mu})$,

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \tag{1}$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda}, \tag{2}$$

$$[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \eta_{\nu\lambda} - \hat{x}_\nu \eta_{\mu\lambda} - i(a_\mu M_{\nu\lambda} - a_\nu M_{\mu\lambda}), \tag{3}$$

where a_μ are components of the deformation vector and $\eta_{\mu\nu} = diag(-1, 1, \cdots, 1)$ defines a spacetime metric. The quantity $a^2 = a_\mu a^\mu$ is Lorentz invariant having a dimension of inverse mass squared, $a^2 \equiv \frac{1}{\kappa^2}$. In the smooth limit $a_\mu \to 0$, the above algebra reduces to a standard one, describing commutative spacetime. It is easy to check that all

Jacobi identities for this algebra are satisfied. Throughout the paper we shall work in units $\hbar = c = 1$.

The symmetry of the deformed spacetime (1) is assumed to be described by an undeformed Poincaré algebra. Thus, in addition to Lorentz generators $M_{\mu\nu}$, we also introduce momenta P_μ which transform as vectors under the Lorentz algebra,

$$[P_\mu, P_\nu] = 0, \tag{4}$$

$$[M_{\mu\nu}, P_\lambda] = \eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu. \tag{5}$$

The algebra (1)-(5), however, does not fix the commutation relation between P_μ and $\hat{x}_\nu$. In fact, there are infinitely many possibilities for the commutation relation between P_μ and $\hat{x}_\nu$, all of which are consistent with the algebra (1)-(5) in a sense that Jacobi identities are satisfied between all generators of the algebra. In this way we have an extended algebra, which includes generators $M_{\mu\nu}$, P_μ and $\hat{x}_\lambda$, and has Jacobi identities satisfied for all combinations of the generators $M_{\mu\nu}$, P_μ and $\hat{x}_\lambda$. Particularly, the algebra generated by P_μ and $\hat{x}_\nu$ is a deformed Heisenberg-Weyl algebra which can generally be written in the form

$$[P_\mu, \hat{x}_\nu] = -i\Phi_{\mu\nu}(P), \tag{6}$$

where $\Phi_{\mu\nu}(P)$ are functions of generators P_μ, satisfying the boundary conditions $\Phi_{\mu\nu}(0) = \eta_{\mu\nu}$ and consistent with relation (1) and Jacobi identities that have to be fulfilled for all combinations of generators $M_{\mu\nu}$, P_μ and $\hat{x}_\lambda$.

The momentum $P_\mu = -i\hat{\partial}_\mu$, expressed in terms of deformed derivative $\hat{\partial}_\mu$, can be realized in a natural way [47] by adopting the identification between deformed and undeformed derivatives, $\hat{\partial}_\mu \equiv \partial_\mu$, implying $P_\mu = -i\partial_\mu$. The deformed algebra (1)-(5) then admits a wide class of realizations

$$\hat{x}_\mu = x^\alpha \Phi_{\alpha\mu}(P), \tag{7}$$

$$M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \tag{8}$$

in terms of the generators of the undeformed Heisenberg algebra,

$$[x_\mu, x_\nu] = 0, \qquad [\partial_\mu, \partial_\nu] = 0, \qquad [\partial_\mu, x_\nu] = \eta_{\mu\nu} \tag{9}$$

and analytic function $\Phi_{\alpha\mu}(P)$, appearing on the right hand side of Eq.(6). By taking this prescription, the deformed algebra (1)-(5) is then automatically satisfied, as well as all Jacobi identities among $\hat{x}_\mu$, $M_{\mu\nu}$, and P_μ. We take one particular realization from the class (7), namely the one of the form

$$\hat{x}_\mu = x_\mu(-ia_\alpha \partial^\alpha + \sqrt{1 - a^2 \partial_\alpha \partial^\alpha}) + i(ax)\partial_\mu. \tag{10}$$

This particular realization of NC spacetime (1) is usually known as the classical Dirac operator representation and has been considered for the first time in Ref.[18]. It is consistent with the deformed algebra (1)-(5) and is also a special case of the whole family of realizations of NC spacetime algebra (1) considered in [70], where it is classified as the Maggiore-type of realizations [56],[57]. In Ref.[47] it is also refered to as *natural realization*. Throughout the paper we shall work only with this type of realization and later on (section 5) with its hermitian form.

With these particular settings, the deformed Heisenberg-Weyl algebra (6) looks as

$$[P_\mu, \hat{x}_\nu] = -i\eta_{\mu\nu}\left(aP + \sqrt{1 + a^2 P^2}\right) + ia_\mu P_\nu. \tag{11}$$

There also exists a universal shift operator Z^{-1} [50] with the following properties:

$$[Z^{-1}, \hat{x}_\mu] = -ia_\mu Z^{-1}, \qquad [Z, P_\mu] = 0, \tag{12}$$

where Z is a regular operator, $ZZ^{-1} = 1$. As an implication of these two equations we have

$$[Z, \hat{x}_\mu] = ia_\mu Z, \qquad \hat{x}_\mu Z \hat{x}_\nu = \hat{x}_\nu Z \hat{x}_\mu. \tag{13}$$

The explicit realization of the universal shift operator Z^{-1} in terms of generators ∂_μ of the Weyl algebra has the form

$$Z^{-1} = -ia_\alpha\partial^\alpha + \sqrt{1 - a^2\partial_\alpha\partial^\alpha}. \tag{14}$$

As a consequence, the Lorentz generators can be expressed in terms of Z as

$$M_{\mu\nu} = i(\hat{x}_\mu P_\nu - \hat{x}_\nu P_\mu)Z, \tag{15}$$

and one can also show the validity of the relation

$$[Z^{-1}, M_{\mu\nu}] = a_\mu P_\nu - a_\nu P_\mu. \tag{16}$$

It is to expect that a deformation of the spacetime structure will affect the algebra of physical fields, leading to a modification of multiplication in the corresponding universal enveloping algebra. Specifically, it means that a spacetime deformation requires a replacement of the usual pointwise multiplication with a deformed product or star product. This will consequently have an impact on physics, particularly it will modify the way in which the field theoretic action should be constructed. It is for this reason that we introduce a star product in a given realization (10). This star product can be introduced in the following way. First we define the unit element 1 as

$$\partial_\mu \triangleright 1 \ = \ 0, \qquad M_{\mu\nu} \triangleright 1 \ = \ 0. \tag{17}$$

This means that Poincaré generators give zero, when acting on unit element 1.

Then, for the particular realization of noncommutative spacetime (1), given by (10), there is a unique map from the algebra $\mathcal{A}_a$ of fields $\phi(x)$ in commutative coordinates x_μ to the enveloping algebra $\hat{\mathcal{A}}_a$ of noncommutative fields $\hat{\phi}(\hat{x})$ in NC coordinates $\hat{x}_\mu$. This map $\Omega : \mathcal{A}_a \longrightarrow \hat{\mathcal{A}}_a$ can uniquely be characterized by

$$\Omega : \ \mathcal{A}_a \longrightarrow \hat{\mathcal{A}}_a, \quad \phi(x) \longmapsto \hat{\phi}(\hat{x}) \quad \text{such that} \quad \hat{\phi}(\hat{x}) \triangleright 1 \ = \ \phi(x). \tag{18}$$

If we further have two fields $\hat{\phi}(\hat{x}), \hat{\psi}(\hat{x})$ in NC coordinates, make their product $\hat{\phi}(\hat{x})\hat{\psi}(\hat{x})$ and ask which field in commutative coordinates this combination belongs to (through the mapping Ω), we arrive at

$$(\phi \star \psi)(x) = \hat{\phi}(\hat{x})\hat{\psi}(\hat{x}) \triangleright 1 \ = \ \hat{\phi}(\hat{x}) \triangleright \psi(x). \tag{19}$$

This prescription defines the star product in any realization and thus specifically defines the star product in the realization (10), too. In Eq.(19) it is assumed that $\hat{\phi}(\hat{x}) \triangleright 1 \ = \ \phi(x)$ and $\hat{\psi}(\hat{x}) \triangleright 1 \ = \ \psi(x)$ and that $\hat{x}$ is given by (10). It is also assumed that $\hat{\phi}(\hat{x})$ and $\hat{\psi}(\hat{x})$ are functions of NC coordinates only and not of derivatives.

In the setting just described, we are considering the fields $\hat{\phi}(\hat{x})$ formed out of polynomials in $\hat{x}$, which constitute an universal enveloping algebra $\hat{\mathcal{A}}_a$ in $\hat{x}$, with the observation that in the algebra $\hat{\mathcal{A}}_a$ the multiplication is given by the standard composition of the operators. In the similar way fields $\phi(x)$ in commutative coordinates are formed out of polynomials in x and thus constitute a vector space of polynomials in x. This vector space can be equipped either with the standard pointwise multiplication or with a deformed multiplication which is comprised within the above defined star product. We shall adequately denote the corresponding algebras with $\mathcal{A}$ and $\mathcal{A}_a$, respectively. This means that the vector space of fields in commutative coordinates, equipped with usual pointwise multiplication, forms a symmetric algebra $\mathcal{A}$, generated by commuting coordinates x_μ, while the same structure, which has a star product as a multiplication operation, constitutes a deformed algebra $\mathcal{A}_a$. While the algebra $\mathcal{A}$ is undeformed and can be considered as a trivial example of an universal enveloping algebra of functions in commutative coordinates, the later one, that is algebra $\mathcal{A}_a$, is deformed one, with deformation being encoded within a star product. Commutative fields $\phi(x)$ that are formed from polynomials in commutative coordinates thus can either be considered as if they belong to an universal enveloping algebra $\mathcal{A}$, generated by commuting coordinates x_μ, or to a deformed algebra $\mathcal{A}_a$, where a deformation is encoded within the star product. In this setting, the unit element 1, having the properties,

$$\phi(x) \triangleright 1 \ = \ \phi(x), \qquad \hat{\phi}(\hat{x}) \triangleright 1 \ = \ \phi(x), \tag{20}$$

$$\partial_\mu \triangleright 1 \quad = \ 0, \qquad M_{\mu\nu} \triangleright 1 \ = \ 0, \tag{21}$$

can be thought of as the unit element in the universal enveloping algebra $\mathcal{A}$ (or more precisely, in a completion of the symmetric algebra $\mathcal{A}$) understood as a module over the deformed Weyl algebra, which is generated by $\hat{x}_\mu$ and ∂_μ, $\mu = 0, 1, ..., n-1$, and allows for infinite series in ∂_μ. It is also understood that NC coordinates $\hat{x}$, appearing in (20), refer to particular realization (10), i.e. they are assumed to be represented by (10).

3. HOPF ALGEBRA STRUCTURE

A deformation of Minkowski spacetime made in accordance with commutation relations (1) (κ-deformation) implies some important questions that have to be addressed. One of them is the question on the real nature of symmetry describing κ-deformed Minkowski space, particularly the question whether the Poincaré symmetry is still a relevant symmetry for theories built on such κ-deformed spaces or instead Poincaré symmetry itself is also affected by deformation. In analysing these issues, we are naturally led to the conclusion that symmetry underlying κ-deformed Minkowski space is deformed too. This κ-deformed Poincaré symmetry can most conveniently be described in terms of quantum Hopf algebra [68]. Besides an algebraic part, which we take by our choice as undeformed and thus described by relations (2),(4) and (5), the full description of κ-deformed Poincaré symmetry also requires an information on the action of Poincaré generators on the enveloping algebra $\hat{\mathcal{A}}_a$, i.e. a type of information which is encoded within the coalgebraic part of the corresponding Hopf algebra. As we shall see, the action of Poincaré generators on deformed Minkowski spacetime and the algebra $\hat{\mathcal{A}}_a$ is deformed. Since deformed action of Poincaré generators leads to a deformed coproduct structure and deformed Leibniz rules, we have as a conclusion that the Hopf algebra, describing κ-deformation of Minkowski space, is characterized by a deformed coalgebraic sector, while simultaneously having an undeformed algebraic part, as discussed above. In this way, the whole deformation is contained within the coalgebraic sector alone.

We now present the basic ingredients of the mathematical structure in question as follows. Denoting κ-Poincaré algebra by $\mathbf{g}$, then as a Lie algebra, it has a unique universal enveloping algebra $\mathcal{U}_a(\mathbf{g})$ which maintains a Lie algebra structure preserved. The algebra $\mathcal{U}_a(\mathbf{g})$ becomes a Hopf algebra if it is endowed with a coalgebra structure, i.e. coproduct $\triangle : \mathcal{U}_a(\mathbf{g}) \longrightarrow \mathcal{U}_a(\mathbf{g}) \otimes \mathcal{U}_a(\mathbf{g})$, counit $\epsilon : \mathcal{U}_a(\mathbf{g}) \longrightarrow \mathbf{C}$ and antipode $S : \mathcal{U}_a(\mathbf{g}) \longrightarrow \mathcal{U}_a(\mathbf{g})$. In this case the algebra $\hat{\mathcal{A}}_a$ can be understood as a module algebra for $\mathcal{U}_a(\mathbf{g})$ since the elements of $\mathcal{U}_a(\mathbf{g})$ act on it by Hopf action.

The algebraic part of the κ-Poincaré Hopf algebra $\mathcal{U}_a(\mathbf{g})$ is given in relations (2),(4) and (5). The coalgebraic part, which includes coproducts for translation ($P_\mu = -i\partial_\mu$), rotation and boost generators [50],[47], is given by the following relations,

$$\triangle \partial_\mu = \partial_\mu \otimes Z^{-1} + \mathbf{1} \otimes \partial_\mu + ia_\mu(\partial_\lambda Z) \otimes \partial^\lambda - \frac{ia_\mu}{2}\Box Z \otimes ia\partial, \tag{22}$$

$$\triangle M_{\mu\nu} = M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu}$$
$$+ ia_\mu\left(\partial^\lambda - \frac{ia^\lambda}{2}\Box\right) Z \otimes M_{\lambda\nu} - ia_\nu\left(\partial^\lambda - \frac{ia^\lambda}{2}\Box\right) Z \otimes M_{\lambda\mu}. \tag{23}$$

In the above expressions Z is the shift operator, Eq.(14), whose copruduct is simply given by

$$\triangle Z = Z \otimes Z. \tag{24}$$

The operator $\Box$ is a deformed d'Alambertian operator [18],[51],[50],

$$\Box = \frac{2}{a^2}(1 - \sqrt{1 - a^2\partial^2}), \tag{25}$$

with the property $\Box \to \partial^2$ as $a \to 0$. We now turn to antipodes for the generators of κ-Poincaré algebra. The antipode for translation generators P_μ can be written in a compact way as

$$S(P_\mu) = \left(-P_\mu - a_\mu P^2 + \frac{1}{2}a_\mu(aP)\Box(P)\right) Z(P), \tag{26}$$

where, in accordance with (14) and (25),

$$Z^{-1}(P) = aP + \sqrt{1 + a^2 P^2}, \qquad \Box(P) = \frac{2}{a^2}(1 - \sqrt{1 + a^2 P^2}). \tag{27}$$

On the other hand, antipode for Lorentz generators has the form

$$S(M_{\mu\nu}) = -M_{\mu\nu} + a_\nu \left(P_\alpha - \frac{a_\alpha}{2}\Box(P) \right) M_{\alpha\mu} - a_\mu \left(P_\alpha - \frac{a_\alpha}{2}\Box(P) \right) M_{\alpha\nu}. \tag{28}$$

Finally counits of the generators of κ-Poincaré algebra remain trivial. One can also check that the coassociativity condition for the coproduct is fulfilled,

$$(id \otimes \triangle)\triangle = (\triangle \otimes id)\triangle, \tag{29}$$

with $id : \mathcal{U}_a(\mathbf{g}) \longrightarrow \mathcal{U}_a(\mathbf{g})$ being an identity operator.

The antipode for translation generators has some usefull properties:

$$i) \quad S(P_\mu)S(P^\mu) = P_\mu P^\mu, \quad \text{or simply} \quad (S(P))^2 = P^2, \tag{30}$$

$$ii) \quad S(S(P_\mu)) = P_\mu, \tag{31}$$

$$iii) \quad Z^{-1}(S(P)) = Z(P), \tag{32}$$

$$iv) \quad \Box(S(P)) = \Box(P). \tag{33}$$

We shall make an extensive use of the above notions and their properties in considerations that will follow shortly and in building the field theory on NC spacetime with noncommutative structure (1).

4. FREE SCALAR FIELD THEORY IN NONHERMITIAN REALIZATION

The realization (10), as it stands is nonhermitian one, i.e. for $\hat{x}_\mu$ in (10) the relation $\hat{x}_\mu^\dagger = \hat{x}_\mu$ does not hold! However, this realization has some convenient and usefull properties [47],[50]. It has been shown in Refs.[69],[70] that for $\hat{x}_\mu$, given in (10), one can write the following identities:

$$e^{iP\hat{x}} \triangleright 1 \ = \ e^{i\mathbf{K}_\mu(P)x^\mu} \tag{34}$$

and

$$e^{iP\hat{x}} \triangleright e^{iQx} \ = \ e^{i\mathbf{P}_\mu(P,Q)x^\mu}, \tag{35}$$

where the unit element 1 is defined in (17), $P\hat{x} = P^\alpha \hat{x}_\alpha$ and quantities $\mathbf{K}_\mu(P)$, $\mathbf{P}_\mu(P,Q)$ have the form

$$\mathbf{P}_\mu(P,Q) \ = \ Q_\mu + \left(P_\mu Z^{-1}(Q) - a_\mu(PQ) \right) \frac{\sinh(aP)}{aP}$$

$$+ \left[\left(P_\mu(aP) - a_\mu P^2 \right) Z^{-1}(Q) + a_\mu(aP)(PQ) \right] \frac{\cosh(aP) - 1}{(aP)^2}, \tag{36}$$

$$\mathbf{K}_\mu(P) = \mathbf{P}_\mu(P,0) \ = \ P_\mu \frac{e^{aP} - 1}{aP} - a_\mu P^2 \frac{\cosh(aP) - 1}{(aP)^2}. \tag{37}$$

As before (see Eq.(27)), the quantity $Z^{-1}(Q)$ can be expressed as

$$Z^{-1}(Q) \ = \ aQ + \sqrt{1 + a^2 Q^2} \ = \ e^{a\mathbf{K}^{-1}(Q)}, \tag{38}$$

where

$$\mathbf{K}_\mu^{-1}(P) = \left[P_\mu - \frac{a_\mu}{2}\Box(P) \right] \frac{\ln\left(Z^{-1}(P) \right)}{Z^{-1}(P) - 1} \tag{39}$$

is the inverse transformation of (37). It is also understood that quantities like (PQ) have the meaning of the scalar product between n-momenta P and Q in a Minkowski space with signature $\eta_{\mu\nu} = diag(-1, 1, \cdots, 1)$ and that n-momenta P and Q allow the identification with the operator $-i\partial$ (e.g. $Q = -i\partial$), as discussed at the beginning.

According to relation (34) and the definition of the star product introduced in (19), we can write for the star product between two plane waves

$$e^{iPx} \star e^{iQx} = e^{i\mathbf{K}^{-1}(P)\hat{x}} \rhd e^{iQx} = e^{i\mathcal{D}_\mu(P,Q)x^\mu}, \tag{40}$$

where, in accordance with (35),

$$\mathcal{D}_\mu(P,Q) = \mathbf{P}_\mu(\mathbf{K}^{-1}(P), Q), \tag{41}$$

with $\mathbf{K}^{-1}(P)$ being given in (39). It can readily be shown that the shift operator $Z^{-1}(P)$ and the d'Alambertian operator $\Box(P)$ can be expressed in terms of the quantity $\mathbf{K}^{-1}(P)$ as

$$Z^{-1}(P) \equiv aP + \sqrt{1 + a^2 P^2} = e^{a\mathbf{K}^{-1}(P)}, \tag{42}$$

$$\Box(P) \equiv \frac{2}{a^2}\left[1 - \sqrt{1 + a^2 P^2}\right] = 2\frac{1 - \cosh\left(a\mathbf{K}^{-1}(P)\right)}{\left(a\mathbf{K}^{-1}(P)\right)^2}(\mathbf{K}^{-1}(P))^2. \tag{43}$$

We can make use of relations (42) and (43) to find the function $\mathcal{D}(P,Q)$ in (41), which determines the momentum addition rule, $\mathcal{D}(P,Q) = P \oplus Q$, in κ-deformed Minkowski space (1). This generalized rule for addition of momenta turns out [50],[47] to have the form

$$\mathcal{D}_\mu(P,Q) = (P \oplus Q)_\mu = P_\mu Z^{-1}(Q) + Q_\mu - a_\mu(PQ)Z(P) + \frac{1}{2}a_\mu(aQ)\Box(P)Z(P). \tag{44}$$

A comparison of Eq.(44) against expression (22), which gives the coproduct for translation generators, $\triangle\partial_\mu$, reveals that it is possible to make an identification

$$i\mathcal{D}_\mu(-i\partial \otimes 1, 1 \otimes (-i\partial)) = \triangle\partial_\mu, \tag{45}$$

showing that function $\mathcal{D}(P,Q) = P \oplus Q$, besides giving the rule for adding momenta, also comprises a deformed Leibniz rule and the corresponding coproduct for translation generators of κ-Poincaré algebra. This result, i.e. Eq.(45), is completely consistent with the general definition of the star product,

$$f \star g = m_\star(f \otimes g) \tag{46}$$

and with the expression describing mutual relationship between star product and coproduct,

$$\partial_\mu(f \star g) = m_\star(\triangle\partial_\mu(f \otimes g)), \tag{47}$$

which is the relation that immediately follows from (46) after taking a derivative on both sides in (46). In the above two expressions $m_\star : \mathcal{A}_a \otimes \mathcal{A}_a \longrightarrow \mathcal{A}_a$ denotes a deformed multiplication in the algebra of commutative and smooth functions. After applying (47) to (40), Eq.(45) follows immediately, which, as we have seen, is the result expected on the ground of comparison between Eqs.(44) and (22). This shows an internal consistency of the entire setting we are working in, since different approaches bring about the same conclusion.

With the known coproduct, it is a straightforward procedure [51],[47] to find a star product between two arbitrary elements f and g in the algebra $\mathcal{A}_a$, generalizing in this way relation (40) that holds for plane waves. Thus, the star product, describing noncommutative features of the spacetime with commutation relations (1), has the form

$$(f \star g)(x) = \lim_{u \to x} m\left(e^{x^\alpha(\triangle - \triangle_0)\partial_\alpha} f(u) \otimes g(u)\right), \tag{48}$$

where $\triangle_0\partial_\mu = \partial_\mu \otimes 1 + 1 \otimes \partial_\mu$, $\triangle\partial_\mu$ is given in (22) and $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ is the undeformed multiplication map in the algebra $\mathcal{A}$, namely, $m(f(x) \otimes g(x)) = f(x)g(x)$. The expression (48) gives a general form for the star product on κ-Minkowski spacetime (1). The star product (48) with the coproduct (22) acts as a deformed multiplication map in the algebra $\mathcal{A}_a$ of smooth, commutative functions. It also reflects the noncommutative nature of κ-Minkowski spacetime and due to coassociativity of the coproduct, Eq.(29), is associative. The opposite is also true, an associativity of star product (48) implies coassociativity of the coproduct, Eq.(29).

In what follows we shall use the results obtained so far in order to build a massive, generally complex, scalar field theory on κ-deformed Minkowski space and to relate it to a corresponding field theory on undeformed Minkowski space. A proper construction of field theory, besides using a star product, i.e. deformed multiplication between the elements of algebra $\mathcal{A}_a$, instead of usual pointwise multiplication between these elements, also requires an introduction of the adjoint (or hermitian conjugation) operation $\dagger$, as an obverse to common complex conjugation operation used in standard field theory. However, at this place it should be emphasized that the adjoint operation $\dagger$ will have different actions when applied to the elements of deformed algebras $\mathcal{A}_a$ and $\hat{\mathcal{A}}_a$, respectively and since we want to establish a full correspondence between these two deformed algebras (through the map Ω), we also have to know the respective actions of the adjoint operation $\dagger$, performed on the corresponding algebras $\mathcal{A}_a$ and $\hat{\mathcal{A}}_a$. Turned in other way, this means that the adjoint operation $\dagger$ induces two different operations $\dagger$, one exhibited when applied to the elements of algebra $\mathcal{A}_a$ of commutative coordinates and the other when applied to the elements of algebra $\hat{\mathcal{A}}_a$ of noncommutative coordinates. In what follows we shall take the same symbol $\dagger$ for both of these operations, but it should be always kept in mind that they are mutually different [1].

In the case that one deals with realization of $\hat{x}$ which is not hermitian, as is the case with realization (10), the construction of scalar field theory cannot be done in a way consistent with prescriptions (18) and (19), because in these circumstances noncommutative plane waves will not be unitary operators and thus it is not clear at all how should adjoint operator of $e^{iP\hat{x}}$ look like and what plane wave in commutative coordinates this adjoint operator would correspond to through the map Ω, Eq.(18). In the next section we shall demonstrate how these type of problems can be overcome. In particular, we shall see how we can achieve that for every adjoint of the usual commutative plane wave, a corresponding adjoint of noncommutative plane wave can be uniquely defined in a way that is consistent with prescriptions (18) and (19).

We take the adjoint of the standard plane wave to be defined as [45],[47]

$$\left(e^{iPx}\right)^\dagger = e^{iS(P)x}, \tag{49}$$

where $S(P)$ is antipode (26) and $\dagger$ means the adjoint operation defined on $\mathcal{A}_a$ (see the footnote 1). In the next section this relation will be justified on the more fundamental ground, but for the moment we just take over it without going into the details. The standard, commutative, generally complex scalar field is assumed to have a Fourier expansion

$$\phi(x) = \int d^n P \ \tilde{\phi}(P) \ e^{iPx}. \tag{50}$$

Now we can write the action for non-interacting complex massive scalar field as

$$\begin{aligned} S[\phi] &= \int d^n x \ \mathcal{L}(\phi, \partial_\mu \phi) \\ &= \frac{1}{2} \int d^n x \ (\partial_\mu \phi)^\dagger \star (\partial^\mu \phi) + \frac{m^2}{2} \int d^n x \ \phi^\dagger \star \phi. \end{aligned} \tag{51}$$

---

[1] The adjoint operation $\dagger$ which acts on the elements of $\hat{\mathcal{A}}_a$ is a standard one and is characterized by the property $P_\mu{}^\dagger = (-i\hat{\partial}_\mu)^\dagger \equiv (-i\partial_\mu)^\dagger = P_\mu$, while the adjoint operation $\dagger$ which acts on the elements of $\mathcal{A}_a$ is characterized by the property $P_\mu{}^\dagger = (-i\partial_\mu)^\dagger = -S(P_\mu)$, with $S(P_\mu)$ beeing the antipode (26). We recall that $\mathcal{A}_a$ is the algebra of commutative functions whose multiplication is being given by the star product. Thus, there should be no problem to distinguish between these two adjoint operations since each of them acts only on $\hat{\mathcal{A}}_a$ or $\mathcal{A}_a$, respectively. Thus, whenever we have $\hat{x}^\dagger$ or $\hat{\phi}^\dagger(\hat{x}^\dagger)$, it is understood that $\dagger$ means the adjoint operation defined on $\hat{\mathcal{A}}_a$, while whenever we have $x^\dagger$ or $\phi^\dagger(x)$, it is understood that $\dagger$ stands for the adjoint operation defined on $\mathcal{A}_a$.

In order to find its relation with the corresponding action on undeformed Minkowski spacetime, we shall consider an integral expression of the form

$$\int d^n x \; \psi^\dagger \star \phi, \tag{52}$$

and rewrite it in terms of pointwise multiplication between arbitrary elements ψ and ϕ in the algebra $\mathcal{A}$. This will result in a very usefull integral mathematical identity that will have a crucial role in our analysis. According to Eqs.(49) and (50) we have a Fourier expansion for the adjoint of the field ϕ,

$$\phi^\dagger(x) = \int d^n P \; \tilde{\phi}^*(P) \; e^{iS(P)x}, \tag{53}$$

where $*$ denotes the standard complex conjugation. Consequently, by anticipating deformed multiplication between plane waves, Eq.(40), it follows

$$
\begin{aligned}
\int d^n x \; \psi^\dagger \star \phi &= \int d^n x \int d^n P \int d^n Q \; \tilde{\psi}^*(P)\tilde{\phi}(Q) \; e^{iS(P)x} \star e^{iQx} \\
&= \int d^n P \int d^n Q \; \tilde{\psi}^*(P)\tilde{\phi}(Q) \int d^n x \; e^{i\mathcal{D}(S(P),Q)x} \\
&= \int\int d^n P d^n Q \; \tilde{\psi}^*(P)\tilde{\phi}(Q)(2\pi)^n \delta^{(n)}\left(\mathcal{D}(S(P),Q)\right),
\end{aligned} \tag{54}
$$

where the quantity in the argument of $\delta^{(n)}$-function can be deduced from (26),(44) and from the properties (30)-(33),

$$
\begin{aligned}
\mathcal{D}_\mu(S(P),Q) =& \left(-P_\mu - a_\mu P^2 + \frac{1}{2}a_\mu(aP)\Box(P)\right)Z(P)Z^{-1}(Q) + Q_\mu + a_\mu(PQ) \\
&+ a_\mu(aQ)P^2 - \frac{1}{2}a_\mu(aQ)(aP)\Box(P) + \frac{1}{2}a_\mu(aQ)\Box(P)Z^{-1}(P).
\end{aligned} \tag{55}
$$

$Z^{-1}(P)$ and $\Box(P)$ above are given by Eqs.(42) and (43), respectively. To calculate $\delta^{(n)}$-function in (54), we use the identity

$$\delta^{(n)}\left(F(P,Q)\right) = \sum_i \frac{\delta^{(n)}(Q - Q_i)}{\left|\det\left(\frac{\partial F_\mu(P,Q)}{\partial Q_\nu}\right)\right|_{Q=Q_i}}, \tag{56}$$

where the expression in denominator is $n \times n$ Jacobi determinant of the transformation $Q \longmapsto F(P,Q)$, with the reminder that while Q is treated as an independent variable, P is assumed to be a parameter. On the other hand Q_i are zeros of the generic function F, which, for the need of our calculation, we specialize in this case to $F(P,Q) \equiv \mathcal{D}(S(P),Q)$. To proceed, we need matrix entries $\frac{\partial \mathcal{D}_\mu(S(P),Q)}{\partial Q^\lambda}$ of Jacobian, which are given by

$$
\begin{aligned}
\frac{\partial \mathcal{D}_\mu(S(P),Q)}{\partial Q^\lambda} =& \left(-P_\mu - a_\mu P^2 + \frac{1}{2}a_\mu(aP)\Box(P)\right)Z(P)\left(a_\lambda + \frac{a^2}{\sqrt{1+a^2Q^2}}Q_\lambda\right) \\
&+ \eta_{\mu\lambda} + a_\mu P_\lambda + a_\mu a_\lambda P^2 + \frac{1}{2}a_\mu a_\lambda \Box(P)\sqrt{1+a^2P^2}.
\end{aligned} \tag{57}
$$

For simplicity and just for the sake of calculation of the determinant, we take the four-vector of the deformation parameter a to be colinear with the time direction, $a = (a_0, 0, ..., 0)$, and write the resulting Jacobian in corresponding Lorentz frame,

$$
\det\left(\frac{\partial \mathcal{D}_\mu(S(P),Q)}{\partial Q^\lambda}\right)_{Q=P} =
\begin{vmatrix}
-c & bZ(P)a_0^2cP_1 + a_0P_1 & \cdots & bZ(P)a_0^2cP_{n-2} + a_0P_{n-2} & bZ(P)a_0^2cP_{n-1} + a_0P_{n-1} \\
-a_0cP_1 & 1 + Z(P)a_0^2cP_1^2 & \cdots & Z(P)a_0^2cP_1P_{n-2} & Z(P)a_0^2cP_1P_{n-1} \\
. & . & \cdots & . & . \\
. & . & \cdots & . & . \\
-a_0cP_{n-3} & Z(P)a_0^2cP_{n-3}P_1 & \cdots & Z(P)a_0^2cP_{n-3}P_{n-2} & Z(P)a_0^2cP_{n-3}P_{n-1} \\
-a_0cP_{n-2} & Z(P)a_0^2cP_{n-2}P_1 & \cdots & 1 + Z(P)a_0^2cP_{n-2}^2 & Z(P)a_0^2cP_{n-2}P_{n-1} \\
-a_0cP_{n-1} & Z(P)a_0^2cP_{n-1}P_1 & \cdots & Z(P)a_0^2cP_{n-1}P_{n-2} & 1 + Z(P)a_0^2cP_{n-1}^2
\end{vmatrix}
\tag{58}
$$

where the quantities c and b are defined as

$$c = \frac{1}{\sqrt{1 - a_0^2 P^2}}, \qquad b = a_0 P^2 + P_0 \sqrt{1 - a_0^2 P^2}. \tag{59}$$

The Jacobian (58) can be calculated to give

$$\det\left(\frac{\partial \mathcal{D}_\mu(S(P), Q)}{\partial Q^\lambda}\right)_{Q=P} = -\frac{1}{\sqrt{1 - a_0^2 P^2}}, \qquad \text{and thus}$$

$$\det\left(\frac{\partial \mathcal{D}_\mu(S(P), Q)}{\partial Q_\lambda}\right)_{Q=P} = -\det\left(\frac{\partial \mathcal{D}_\mu(S(P), Q)}{\partial Q^\lambda}\right)_{Q=P} = \frac{1}{\sqrt{1 + a^2 P^2}}. \tag{60}$$

Due to the fact that all quantities are written in a covariant fashion, we can do the same with the final result for the determinant (58) and rewrite it also in a covariant form, giving rise to the last line in (60). In fact, all we had to do was to rewrite a_0^2 in a covariant form, $a_0^2 = -a^2$, respecting the signature we work with. As a consequence of the definition of the antipode, $\mathcal{D}(S(P), P) = \mathcal{D}(P, S(P)) = 0$, there is only one zero for the expression in (55). This zero is the trivial one, $Q = P$, and was used in calculating the Jacobian (58). Now we have

$$\delta^{(n)}\left(\mathcal{D}(S(P), Q)\right) = \sqrt{1 + a^2 P^2}\, \delta^{(n)}(Q - P). \tag{61}$$

By plugging this into (54) and using $P_\mu = -i\partial_\mu$, as well as the integral representation of $\delta^{(n)}$-function,

$$\delta^{(n)}(Q - P) = \frac{1}{(2\pi)^n} \int d^n x\, e^{i(Q-P)x}, \tag{62}$$

we get the following integral identity

$$\int d^n x\, \psi^\dagger \star \phi = \int d^n x\, \psi^*(x) \sqrt{1 - a^2 \partial^2}\, \phi(x), \tag{63}$$

where the symbol $*$ denotes the usual complex conjugation in the undeformed algebra $\mathcal{A}$. When the identity (63) is applied to the action (51), we get

$$S[\phi] = \frac{1}{2} \int d^n x\, \left[(\partial_\mu \phi)^*(x) \sqrt{1 - a^2 \partial^2}\, (\partial^\mu \phi)(x) + m^2 \phi^*(x) \sqrt{1 - a^2 \partial^2}\, \phi(x)\right], \tag{64}$$

which is the action that describes a nonlocal, relativistically invariant free scalar field theory on undeformed Minkowski spacetime [45],[47]. The action (64) is also translationally invariant, since the star product (40), i.e. (48) is translationally invariant. This can indeed be verified by utilising that κ-Minkowski space is invariant [15],[48] under translations $\hat{x}_\mu \longrightarrow \hat{x}_\mu + \hat{y}_\mu$ if

$$[\hat{y}_\mu, \hat{y}_\nu] = i(a_\mu \hat{y}_\nu - a_\nu \hat{y}_\mu), \qquad [\hat{x}_\mu, \hat{y}_\nu] = 0. \tag{65}$$

Another point that we anticipate is that in terms of realizations, on which our approach is based, we can write

$$\hat{x}_\mu = x^\alpha \Phi_{\alpha\mu}(\partial_x), \qquad \hat{y}_\mu = y^\alpha \Phi_{\alpha\mu}(\partial_y), \tag{66}$$

where ∂_x and ∂_y are derivatives with respect to different sets of commutative coordinates x and y, respectively. Now, by taking into account relations

$$e^{i\mathbf{K}^{-1}(P)\hat{x}} \triangleright 1 = e^{iPx}, \qquad e^{i\mathbf{K}^{-1}(Q)\hat{x}} \triangleright 1 = e^{iQx}, \tag{67}$$

and by making use of (65), we have

$$e^{i\mathbf{K}^{-1}(P)(\hat{x}+\hat{y})} \triangleright 1 = e^{i\mathbf{K}^{-1}(P)\hat{x}} e^{i\mathbf{K}^{-1}(P)\hat{y}} \triangleright 1 = e^{iPy} e^{i\mathbf{K}^{-1}(P)\hat{x}} \triangleright 1 = e^{iP(x+y)}, \tag{68}$$

and similarly

$$e^{i\mathbf{K}^{-1}(Q)(\hat{x}+\hat{y})} \triangleright 1 = e^{iQ(x+y)}. \tag{69}$$

Consequently we can write

$$
\begin{aligned}
e^{iP(x+y)} \star e^{iQ(x+y)} &= e^{i\mathbf{K}^{-1}(P)(\hat{x}+\hat{y})} e^{i\mathbf{K}^{-1}(Q)(\hat{x}+\hat{y})} \triangleright 1 \\
&= e^{i\mathcal{D}(P,Q)x} e^{i\mathcal{D}(P,Q)x} = e^{i\mathcal{D}(P,Q)(x+y)},
\end{aligned}
\tag{70}
$$

showing that the star product in nonhermitian realization is translationally invariant and so is the action (64).

5. FREE SCALAR FIELD THEORY IN HERMITIAN REALIZATION

The aim of this section is to finally resolve the open problems that are addressed, but not in any case answered in the previous section. It specifically applies to a problem of proper definition of the adjoint operation †, which would be consistent with the map Ω between algebra $\mathcal{A}_a$ of commutative functions and enveloping algebra $\hat{\mathcal{A}}_a$ of noncommutative functions and with the definition (19) of the star product, as well as with the whole construction of the field theoretic action. In the previous section we have worked with realization (10) which is not hermitian. However, it can be made hermitian by forming the combination

$$
\hat{x}_\mu^h = \frac{1}{2}(\hat{x}_\mu + \hat{x}_\mu^\dagger),
\tag{71}
$$

where † means the adjoint operation defined on $\hat{\mathcal{A}}_a$ (this operation is induced by the standard hermitian conjugation in $\mathcal{A}$, i.e. $(\partial_\mu)^\dagger = -\partial_\mu$, $x_\mu{}^\dagger = x_\mu$, see the footnote 1). This leads to realization

$$
\begin{aligned}
\hat{x}_\mu^h &= \hat{x}_\mu - \frac{1}{2}\frac{a^2\partial_\mu}{\sqrt{1-a^2\partial^2}} \\
&= x_\mu(-ia_\alpha\partial^\alpha + \sqrt{1-a^2\partial_\alpha\partial^\alpha}) + i(ax)\partial_\mu - \frac{1}{2}\frac{a^2\partial_\mu}{\sqrt{1-a^2\partial^2}},
\end{aligned}
\tag{72}
$$

which is hermitian by construction, $(\hat{x}_\mu^h)^\dagger = \hat{x}_\mu^h$, and satisfies Eq.(1), i.e.

$$
[\hat{x}_\mu^h, \hat{x}_\nu^h] = i(a_\mu\hat{x}_\nu^h - a_\nu\hat{x}_\mu^h),
\tag{73}
$$

as well as other commutation relations, Eqs.(2)-(6). Thus, all commutation relations and particularly (3) and (6) remain unchanged, when expressed in terms of $\hat{x}_\mu^h$. The analysis carried out on the basis of hermitian realization (72) will also preserve all the results obtained so far, particularly it will not change the relations (34) and (35), except only for the modification introduced by the additional factors, $A(P)$ and $\tilde{A}(P,Q)$,

$$
e^{iP\hat{x}^h} \triangleright 1 = e^{i\mathbf{K}_\mu(P)x^\mu} A(P)
\tag{74}
$$

and

$$
e^{iP\hat{x}^h} \triangleright e^{iQx} = e^{i\mathbf{P}_\mu(P,Q)x^\mu} \tilde{A}(P,Q),
\tag{75}
$$

where the unit element 1 is defined in (17) and the quantities $\mathbf{P}_\mu(P,Q)$ and $\mathbf{K}(P)$ are given as before, Eqs.(36) and (37), respectively. These last two relations can be inverted and rewritten as

$$
\begin{aligned}
e^{i\mathbf{K}^{-1}(P)\hat{x}^h} \triangleright 1 &= e^{iP_\mu x^\mu} A(\mathbf{K}^{-1}(P)) \\
&\equiv e^{iPx} \sqrt[4]{1+a^2P^2}
\end{aligned}
\tag{76}
$$

and

$$
\begin{aligned}
e^{i\mathbf{K}^{-1}(P)\hat{x}^h} \triangleright e^{iQx} &= e^{i\mathbf{P}_\mu(\mathbf{K}^{-1}(P),Q)x^\mu} \tilde{A}(\mathbf{K}^{-1}(P),Q) \\
&\equiv e^{i\mathcal{D}(P,Q)x} \frac{\sqrt[4]{1+a^2(\mathcal{D}(P,Q))^2}}{\sqrt[4]{1+a^2Q^2}},
\end{aligned}
\tag{77}
$$

where $\mathbf{K}^{-1}(P)$ and $\mathcal{D}(P,Q)$ are given in (39) and (44), respectively, with the same relationship between $\mathbf{P}(\mathbf{K}^{-1}(P),Q)$ and $\mathcal{D}(P,Q)$ as in (41). In the above identities, last lines give the form of the additional factors,

$$A(\mathbf{K}^{-1}(P)) = \sqrt[4]{1+a^2 P^2}, \tag{78}$$

$$\tilde{A}(\mathbf{K}^{-1}(P),Q) = \frac{\sqrt[4]{1+a^2(\mathcal{D}(P,Q))^2}}{\sqrt[4]{1+a^2 Q^2}}. \tag{79}$$

As is the case with the identities (34) and (35) which are rigorously established for the realization (10), the identities in Eqs.(76) and (77) are also rigorously established, but for the realization $\hat{x}^h$, Eq.(72). They can relatively easily be checked up to second order in the deformation parameter a. To do this, one has to insert (39) and (72) into l.h.s. of (76) (or (77)), make Taylor expansion and gather all terms up to second order in a.

On the basis of these results, we can introduce noncommutative plane waves $\hat{e}_P^+$ with label $+$ as follows

$$\hat{e}_P^+ \equiv \frac{e^{i\mathbf{K}^{-1}(P)\hat{x}^h}}{\sqrt[4]{1+a^2 P^2}}. \tag{80}$$

This allows us to dissociate a general noncommutative field $\hat{\phi}(\hat{x}^h)$ into elementary Fourier components that are represented by plane waves (80),

$$\hat{\phi}(\hat{x}^h) = \int d^n P\ \tilde{\phi}(P)\ \hat{e}_P^+ = \int d^n P\ \tilde{\phi}(P)\ \frac{e^{i\mathbf{K}^{-1}(P)\hat{x}^h}}{\sqrt[4]{1+a^2 P^2}}. \tag{81}$$

The above expansion is consistent with the Ω-map (18) and with the expansion (50) that can be applied to a general element of the algebra $\mathcal{A}_a$, i.e. to an arbitrary commutative field $\phi(x)$,

$$\hat{\phi}(\hat{x}^h) \triangleright 1 = \int d^n P\ \tilde{\phi}(P)\ \frac{e^{i\mathbf{K}^{-1}(P)\hat{x}^h}}{\sqrt[4]{1+a^2 P^2}} \triangleright 1 = \int d^n P\ \tilde{\phi}(P)\ e^{iPx} = \phi(x). \tag{82}$$

One can further show that the antipode $S(P)$ has the following property

$$\mathbf{K}^{-1}(S(P)) = -\mathbf{K}^{-1}(P). \tag{83}$$

This property will show up as the missing link required for introducing an adjoint operation $\dagger$ in a proper and consistent way and for making a correct and complete correspondence between algebras $\mathcal{A}_a$ and $\hat{\mathcal{A}}_a$. In particular, it allows us to make one-to-one correspondence between hermitian conjugated noncommutative plane waves and hermitian conjugated commutative plane waves. Thus, it is crucial in building a field theory on κ-Minkowski spacetime in an internally consistent way. Due to the fact that realization $\hat{x}^h$, Eq.(72), is hermitian, the plane wave (80) will be an unitary operator, so that corresponding hermitian conjugation will lead to

$$\frac{\left(e^{i\mathbf{K}^{-1}(P)\hat{x}^h}\right)^\dagger}{\sqrt[4]{1+a^2 P^2}} = \frac{e^{-i\mathbf{K}^{-1}(P)\hat{x}^h}}{\sqrt[4]{1+a^2 P^2}} = \frac{e^{i\mathbf{K}^{-1}(S(P))\hat{x}^h}}{\sqrt[4]{1+a^2 P^2}}, \tag{84}$$

where in the final step use has been made of the property (83). The stated result enables us to introduce noncommutative plane waves $\hat{e}_P^-$ with label $-$,

$$\hat{e}_P^- \equiv (\hat{e}_P^+)^\dagger = \frac{e^{i\mathbf{K}^{-1}(S(P))\hat{x}^h}}{\sqrt[4]{1+a^2 P^2}}. \tag{85}$$

It is worthy to note that due to the property (30) of the antipode, the noncommutative plane wave $\hat{e}_P^-$ can be identified with the noncommutative plane wave $\hat{e}_{S(P)}^+$ with label $+$, whose momentum is obtained by the transformation $P \longmapsto S(P)$. Thus, we have $\hat{e}_P^- = \hat{e}_{S(P)}^+$.

According to (76) and the property (30) of the antipode, the plane waves (80) and (85) act on the unit element 1 as

$$\hat{e}_P^+ \, \triangleright 1 \; = \; e^{iPx}, \qquad \hat{e}_P^- \, \triangleright 1 \; = \; e^{iS(P)x}, \tag{86}$$

realizing in this way the following one-to-one correspondence

$$\Omega: \; \mathcal{A}_a \longrightarrow \hat{\mathcal{A}}_a \qquad \text{with the property} \qquad \begin{aligned} e^{iPx} &\longmapsto \hat{e}_P^+, \\ e^{iS(P)x} &\longmapsto \hat{e}_P^-. \end{aligned} \tag{87}$$

With these correlations we are now as well in a position to draw a definite and unique correspondence between hermitian conjugated elements of the algebras $\mathcal{A}_a$ and $\hat{\mathcal{A}}_a$. In particular, we can take an adjoint of the expansion (81) by applying $\dagger$ to it and then, by utilising Eq.(76), act upon the unit element (17) to obtain

$$\hat{\phi}^\dagger(\hat{x}^h) \triangleright 1 \; = \; \int d^n P \; \tilde{\phi}^*(P) \, \frac{e^{i\mathbf{K}^{-1}(S(P))\hat{x}^h}}{\sqrt[4]{1 + a^2 P^2}} \triangleright 1 \; = \; \int d^n P \; \tilde{\phi}^*(P) \, e^{iS(P)x} \; = \; \phi^\dagger(x), \tag{88}$$

in accordance with (53). This makes the Ω-map, $\Omega: \mathcal{A}_a \longrightarrow \hat{\mathcal{A}}_a$, fully characterized now in a sense that we now also know how to make a correspondence between hermitian conjugated elements of the algebras $\mathcal{A}_a$ and $\hat{\mathcal{A}}_a$, namely,

$$\Omega: \mathcal{A}_a \longrightarrow \hat{\mathcal{A}}_a, \qquad \phi^\dagger(x) \longmapsto \hat{\phi}^\dagger(\hat{x}^h), \qquad \text{such that} \qquad \hat{\phi}^\dagger(\hat{x}^h) \, \triangleright 1 \; = \; \phi^\dagger(x). \tag{89}$$

It is important to note that this correspondance is possible only for the hermitian realization, $\hat{x}^h$, $(\hat{x}^h)^\dagger = \hat{x}^h$, because the notion of the adjoint operation $\dagger$ has a sense in that case. In other words, it is only in that case when the hermitian representation of noncommutative coordinates is used that the adjoint operation $\dagger$ can be introduced in a consistent way.

The one-to-one correspondence described above directly explaines and justifies why the prescription (49), made in the previous section, is correct and in agreement with the Ω-map, Eq.(18), which provides the communication between two different descriptions of the same physics, one in terms of ordinary, commutative fields and coordinates and the other in terms of noncommutative coordinates and fields. This communication between two descriptions, while realized through the isomorphism Ω, was still not fully specified all until the moment when it became clear how this correspondence should look like in the case of hermitian conjugated fields. Up to that moment, the prescription for mapping noncommutative fields into commutative ones was generally known, but it was not known what would hermitian conjugated noncommutative field transform into under the Ω-map and how would the hermitian conjugation (adjoint operation) look like at all. To solve these ambiguities, it was first necessary to introduce the notion of realization which is hermitian, so that noncommutative plane wave can be treated as an unitary operator. The second important point was to realize that there exists identity of the form (83), whose significance shows up as a crucial one in the construction process. These favourable circumstances enabled the complete specification of the isomorphic Ω-map, fixed up all problems and inconsistencies that have existed before and encompassed the whole picture in a neat way.

It is now possible to introduce a star product (19) corresponding to hermitian realization $\hat{x}^h$, Eq.(72). We designate it with $\star_h$. According to the general definition (19) of the star product, we can write

$$\begin{aligned} e^{iPx} \star_h e^{iQx} &= \frac{e^{i\mathbf{K}^{-1}(P)\hat{x}^h}}{\sqrt[4]{1 + a^2 P^2}} \frac{e^{i\mathbf{K}^{-1}(Q)\hat{x}^h}}{\sqrt[4]{1 + a^2 Q^2}} \; \triangleright 1 \\ &= e^{i\mathcal{D}(P,Q)x} \frac{\sqrt[4]{1 + a^2(\mathcal{D}(P,Q))^2}}{\sqrt[4]{1 + a^2 P^2} \; \sqrt[4]{1 + a^2 Q^2}}, \end{aligned} \tag{90}$$

where we have successively applied identities (74) and (75). The corresponding star product between arbitrary two elements f and g of the algebra $\mathcal{A}_a$ modifies accordingly,

$$(f \star_h g)(x) = \lim_{u \to x} m \left(e^{x^\alpha(\triangle - \triangle_0)\partial_\alpha} \sqrt[4]{\frac{1 - a^2 \triangle(\partial^2)}{(1 - a^2 \, \partial^2 \otimes 1)(1 - a^2 \, 1 \otimes \partial^2)}} \, f(u) \otimes g(u) \right), \tag{91}$$

where the homomorphic property of the coproduct $\triangle(\partial_\alpha)$, Eq.(22), is utilised, namely, $\triangle(\partial_\alpha)\triangle(\partial^\alpha) = \triangle(\partial^2)$. In this way, the nonhermitian version of the star product, Eq.(48), is replaced by the hermitian one, Eq.(91).

It can be noted that unlike the star product (48), the star product (91), corresponding to hermitian realization $((\hat{x}^h)^\dagger = \hat{x}^h)$, breaks translational invariance. This can be seen by following the same steps that result with relation (70). One only has to take care of the cosequences of the hermitization process. These imply that instead of relations (67), one has to deal with the following ones

$$e^{i\mathbf{K}^{-1}(P)\hat{x}^h} \triangleright 1 \;=\; A(\mathbf{K}^{-1}(P))e^{iPx}, \qquad e^{i\mathbf{K}^{-1}(Q)\hat{x}^h} \triangleright 1 \;=\; A(\mathbf{K}^{-1}(Q))e^{iQx}, \tag{92}$$

where $A(\mathbf{K}^{-1}(P))$ and $\tilde{A}(\mathbf{K}^{-1}(P),Q)$ in the expression below are determined by Eqs.(78) and (79), respectively. Having that, translation $\hat{x}_\mu \longrightarrow \hat{x}_\mu + \hat{y}_\mu$ leads to

$$
\begin{aligned}
e^{iP(x+y)} \star_h e^{iQ(x+y)} &= \frac{e^{i\mathbf{K}^{-1}(P)(\hat{x}^h+\hat{y}^h)}}{\left[A(\mathbf{K}^{-1}(P))\right]^2} \frac{e^{i\mathbf{K}^{-1}(Q)(\hat{x}^h+\hat{y}^h)}}{\left[A(\mathbf{K}^{-1}(Q))\right]^2} \triangleright 1 \\
&= \frac{1}{\left[A(\mathbf{K}^{-1}(P))\right]^2} \frac{1}{A(\mathbf{K}^{-1}(Q))} e^{i\mathbf{K}^{-1}(P)\hat{y}^h} e^{i\mathbf{K}^{-1}(Q)\hat{y}^h} \triangleright e^{i\mathcal{D}(P,Q)x} \tilde{A}(\mathbf{K}^{-1}(P),Q) \\
&= \left[\frac{\tilde{A}(\mathbf{K}^{-1}(P),Q)}{A(\mathbf{K}^{-1}(P))}\right]^2 e^{i\mathcal{D}(P,Q)(x+y)},
\end{aligned} \tag{93}
$$

leading to a conclusion that the star product (91) is not translation invariant. This is due to the appearance of an extra factor on r.h.s. of Eq.(93). However, due to some other properties of the new star product (91), the free scalar field action, that is constructed in terms of this new star product, will nevertheless be translationally invariant. We shall further discuss the issues related with translational invariance in the concluding section. For the moment let us focus our attention on the other important properties of the new star product.

With this aim, it is interesting to note that noncommutative plane waves with opposite labels (one $+$, the other $-$ and vice versa) are orthonormal among themselves,

$$
\begin{aligned}
\int d^n x \, \hat{e}_P^- \hat{e}_Q^+ \triangleright 1 &= \int d^n x \, \frac{e^{(i\mathbf{K}^{-1}(P)\hat{x}^h)^\dagger}}{\sqrt[4]{1+a^2P^2}} \frac{e^{i\mathbf{K}^{-1}(Q)\hat{x}^h}}{\sqrt[4]{1+a^2Q^2}} \triangleright 1 \\
&= \int d^n x \, \frac{e^{i\mathbf{K}^{-1}(S(P))\hat{x}^h}}{\sqrt[4]{1+a^2P^2}} e^{iQx} = \int d^n x \, \frac{\tilde{A}(\mathbf{K}^{-1}(S(P)),Q)}{\sqrt[4]{1+a^2P^2}} e^{i\mathcal{D}(S(P),Q)x} \\
&= \frac{\tilde{A}(\mathbf{K}^{-1}(S(P)),Q)}{\sqrt[4]{1+a^2P^2}} (2\pi)^n \delta^{(n)}(\mathcal{D}(S(P),Q)) = \frac{\tilde{A}(\mathbf{K}^{-1}(S(P)),Q)}{\sqrt[4]{1+a^2P^2}} (2\pi)^n \sqrt{1+a^2P^2} \, \delta^{(n)}(P-Q) \\
&= \frac{\sqrt[4]{1+a^2(\mathcal{D}(S(P),Q))^2}}{\sqrt[4]{1+a^2P^2} \sqrt[4]{1+a^2Q^2}} (2\pi)^n \sqrt{1+a^2P^2} \, \delta^{(n)}(P-Q) \;=\; (2\pi)^n \, \delta^{(n)}(P-Q).
\end{aligned} \tag{94}
$$

In the second line here, Eqs.(77) and (83) have been used, while in the third line the result for the $\delta^{(n)}$-function (61) was used. The expression considered will be different from zero only for $Q = P$ and since in this case we have $\mathcal{D}(S(Q),Q) = 0$, due to the very definition of the antipode $S(P)$, all factors in the last line of (94) will cancel each other, leading to the orthonormality property of noncommutative plane waves with opposite labels. On the basis of the definition (19) and correspondence (87), the integral in (94) can be recognized and rewritten in terms of the star product, Eq.(90), as follows

$$
\begin{aligned}
\int d^n x \, \hat{e}_P^- \hat{e}_Q^+ \triangleright 1 &= \int d^n x \, \frac{e^{(i\mathbf{K}^{-1}(P)\hat{x}^h)^\dagger}}{\sqrt[4]{1+a^2P^2}} \frac{e^{i\mathbf{K}^{-1}(Q)\hat{x}^h}}{\sqrt[4]{1+a^2Q^2}} \triangleright 1 \\
&= \int d^n x \, e^{iS(P)x} \star_h e^{iQx},
\end{aligned} \tag{95}
$$

which, according to (94), is equal to $(2\pi)^n \delta^{(n)}(P-Q)$. Note the difference of this result when compared to the similar one obtained in the previous section for the star product in nonhermitian realization. In that case the plane waves were not orthonormal to each other and the result was not so simple. By taking an advantage of the result (94), we may now reconsider the expression (52),

$$\int d^n x \; \psi^\dagger \star_h \phi, \tag{96}$$

but this time with the star product $\star_h$ corresponding to hermitian realization (72) and rederive the corresponding mathematical identity as in the previous section and see if it is modified with respect to the identity (63) obtained there. Thus, in accordance with Eqs.(49),(50) and (53) and by making use of the just derived result (94), i.e. the result

$$\int d^n x \; e^{iS(P)x} \star_h e^{iQx} \;=\; (2\pi)^n \; \delta^{(n)}(P-Q), \tag{97}$$

we easily find

$$\int d^n x \; \psi^\dagger \star_h \phi \;=\; \int d^n P \int d^n Q \; \tilde{\psi}^*(P)\tilde{\phi}(Q) \int d^n x \; e^{iS(P)x} \star_h e^{iQx}$$
$$=\; \int d^n P \int d^n Q \; \tilde{\psi}^*(P)\tilde{\phi}(Q)(2\pi)^n \; \delta^{(n)}(P-Q). \tag{98}$$

Utilising further the representation (62) for $\delta^{(n)}$-function, we finally get

$$\int d^n x \; \psi^\dagger \star_h \phi \;=\; \int d^n x \; \psi^*(x)\phi(x), \tag{99}$$

where $*$ denotes the standard complex conjugation operation in the undeformed algebra $\mathcal{A}$. Thus we get a result that the star product for κ-Minkowski space, in a hermitian realization (72), can be replaced with the ordinary pointwise multiplication, under the integration sign. This property is known to hold for canonical type of noncommutativity, as for example for the Groenewold-Moyal plane, but up to this moment, was not established and recognized in the context of κ-Minkowski space.

The fact that under the integration sign we can simply drop out the star product and replace it with an ordinary pointwise multiplication has for consequence that the free massive scalar field action on κ-deformed Minkowski space loses a nonlocal character and takes on an undeformed shape, at least in the part with the mass term. However, before being able to give the final form to the action (51), with star product $\star$ replaced by $\star_h$, we have to know how star product of vector fields behaves upon integration. In other words, we have to know the right form for the partial integration formula valid on κ-Minkowski spacetime and applied to the star product (91). By going through the same steps and by following the similar lines of reasoning which brought us to the result (99), we get the required partial integration formula in the form

$$\int d^n x \; (\partial_\mu \psi)^\dagger \star_h \phi \;=\; -\int d^n x \; \psi^\dagger \star_h \partial_\mu \phi \;=\; -\int d^n x \; \psi^*(x)\partial_\mu \phi(x). \tag{100}$$

A variant of this formula, when derivative is not affected by the adjoint operation, looks as

$$\int d^n x \; \partial_\mu \psi^\dagger \star_h \phi \;=\; \int d^n x \; \psi^\dagger \star_h S(\partial_\mu)\phi \;=\; \int d^n x \; \psi^*(x)S(\partial_\mu)\phi(x), \tag{101}$$

with $S(-i\partial_\mu) = -iS(\partial_\mu)$ being the antipode (26) for translation generators. Having properties (99) and (100) of the star product (91), we are now in a position to obtain a final form for the free massive scalar field action,

$$S[\phi] \;=\; \frac{1}{2} \int d^n x \; (\partial_\mu \phi)^\dagger \star_h \partial^\mu \phi + \frac{m^2}{2} \int d^n x \; \phi^\dagger \star_h \phi$$
$$=\; -\frac{1}{2} \int d^n x \; \phi^*(x) \; \partial_\mu \partial^\mu \phi(x) + \frac{m^2}{2} \int d^n x \; \phi^*(x)\phi(x)$$
$$=\; \frac{1}{2} \int d^n x \; \left[(\partial_\mu \phi)^*(\partial^\mu \phi)(x) + m^2 \phi^*(x)\phi(x) \right], \tag{102}$$

showing that the equivalent theory on undeformed space is no more nonlocal and actually keeps the same form as the initial noncommutative theory. This result is an immediate consequence of properties of the star product (91) corresponding to hermitian realization (72). Due to property (99), the action (102) appears to be translationally invariant, despite being written in terms of the star product that breaks translation invariance. It should also be noted that the form of Klein-Gordon operator in the final expression for the action is usual, undeformed one, $-\partial_\mu \partial^\mu + m^2$. This is in contrast with the analysis carried out in Refs.[18] and [71] where a deformed Klein-Gordon operator $\frac{2}{a^2}(1 - \sqrt{1 - a^2 \partial^2}) + m^2$ was used.

In the case that the original action is defined as

$$S[\phi] \;=\; \frac{1}{2} \int d^n x \; \partial_\mu \phi^\dagger \star_h \partial^\mu \phi + \frac{m^2}{2} \int d^n x \; \phi^\dagger \star_h \phi, \tag{103}$$

the transformation according to partial integration formula (101) would lead to

$$\begin{aligned}
S[\phi] \;&=\; \frac{1}{2} \int d^n x \; \phi^\dagger \star_h S(\partial_\mu) \partial^\mu \phi + \frac{m^2}{2} \int d^n x \; \phi^\dagger \star_h \phi \\
&=\; \frac{1}{2} \int d^n x \left[\phi^*(x) \; S(\partial_\mu) \partial^\mu \phi(x) + m^2 \phi^*(x) \phi(x) \right],
\end{aligned} \tag{104}$$

and the final form for the action would represent an equivalent free massive scalar field theory on ordinary Minkowski spacetime, with modified Klein-Gordon operator, $S(\partial_\mu) \partial^\mu + m^2$.

In order to discuss the issue of reality of the scalar field $\phi(x)$, let us see what is happening if we change the variables of integration in momentum space according to $P \longmapsto S(P)$. The meassure in momentum space will then transform according to

$$d^n S(P) \;=\; \det\left(\frac{\partial S(P_\mu)}{\partial P_\nu} \right) d^n P. \tag{105}$$

The Jacobian in (105) needs to be found and in order to do this, in the same way as we have already done it before to calculate Jacobian (58), we orient deformation four-vector a to point along the time-direction, $a = (a_0, 0, ..., 0)$, in which case the required matrix entries of the corresponding Jacobian of transformation look as

$$\begin{aligned}
\frac{\partial S(P_0)}{\partial P_0} \;&=\; -1 - a_0^2 \vec{P}^2 \frac{Z(P)}{\sqrt{1 - a_0^2 P^2}}, \\
\frac{\partial S(P_0)}{\partial P_i} \;&=\; -a_0 P_i Z(P) \left(2 + \frac{a_0^2 \vec{P}^2}{\sqrt{1 - a_0^2 P^2}} Z(P) \right), \\
\frac{\partial S(P_i)}{\partial P_0} \;&=\; -P_i \frac{a_0}{\sqrt{1 - a_0^2 P^2}} Z(P), \\
\frac{\partial S(P_i)}{\partial P_j} \;&=\; -Z(P) \left(\delta_{ij} + a_0^2 \frac{P_i P_j}{\sqrt{1 - a_0^2 P^2}} Z(P) \right).
\end{aligned} \tag{106}$$

Thus, the Jacobian alone is given by

$$\det\left(\frac{\partial S(P_\mu)}{\partial P_\nu} \right) = \begin{vmatrix}
-1 - a_0 \vec{P}^2 \tilde{b} & -a_0 P_1 Z(P)\left(2 + a_0 \vec{P}^2 \tilde{b}\right) & \cdots & -a_0 P_{n-2} Z(P)\left(2 + a_0 \vec{P}^2 \tilde{b}\right) & -a_0 P_{n-1} Z(P)\left(2 + a_0 \vec{P}^2 \tilde{b}\right) \\
-\tilde{b} P_1 & -Z(P)\left(1 + a_0 \tilde{b} P_1^2\right) & \cdots & -a_0 \tilde{b} Z(P) P_1 P_{n-2} & -a_0 \tilde{b} Z(P) P_1 P_{n-1} \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
-\tilde{b} P_{n-3} & -a_0 \tilde{b} Z(P) P_{n-3} P_1 & \cdots & -a_0 \tilde{b} Z(P) P_{n-3} P_{n-2} & -a_0 \tilde{b} Z(P) P_{n-3} P_{n-1} \\
-\tilde{b} P_{n-2} & -a_0 \tilde{b} Z(P) P_{n-2} P_1 & \cdots & -Z(P)\left(1 + a_0 \tilde{b} P_{n-2}^2\right) & -a_0 \tilde{b} Z(P) P_{n-2} P_{n-1} \\
-\tilde{b} P_{n-1} & -a_0 \tilde{b} Z(P) P_{n-1} P_1 & \cdots & -a_0 \tilde{b} Z(P) P_{n-1} P_{n-2} & -Z(P)\left(1 + a_0 \tilde{b} P_{n-1}^2\right)
\end{vmatrix} \tag{107}$$

where the quantity $\tilde{b}$ is defined as

$$\tilde{b} = \frac{a_0}{\sqrt{1 - a_0^2 P^2}} Z(P). \tag{108}$$

The Jacobian (107) can be calculated to give

$$d^n S(P) = Z^{n-1}(P) d^n P, \tag{109}$$

with $Z(P)$ given in (42).

The commutative scalar field $\phi(x)$ will be real if it satisfies the condition

$$\phi^\dagger(x) = \phi(x). \tag{110}$$

From this condition, from the expansions (50) and (53) and from Eq.(109), we find that the reality condition (110) imposed on the scalar field,

$$\phi^\dagger(x) = \int d^n P \, \tilde{\phi}^*(P) \, e^{iS(P)x}$$
$$= \int d^n S(P) \, Z^{-(n-1)}(P) \, \tilde{\phi}^*(P) \, e^{iS(P)x} = \int d^n S(P) \, \tilde{\phi}(S(P)) \, e^{iS(P)x} = \phi(x), \tag{111}$$

leads to the following condition involving its Fourier components,

$$\tilde{\phi}(S(P)) = Z^{-(n-1)}(P) \tilde{\phi}^*(P). \tag{112}$$

The same condition would emerge if we imposed the reality condition on the noncommutative fields, i.e. $\hat{\phi}^\dagger(\hat{x}^h) = \hat{\phi}(\hat{x}^h)$.

6.   CONCLUDING REMARKS AND DISCUSSION

We now turn again to discussion of the properties of the star product (91) corresponding to hermitian realization (72). With this purpose, the identity (99) can successively be applied twice to get

$$\int d^n x \, \psi^\dagger \star_h \phi = \int d^n x \, (\phi^*(x))^* \psi^*(x) = \int d^n x \, \phi^{*\dagger} \star_h \psi^*, \tag{113}$$

showing that the star product (91) has a generalized trace property. We point out that in the approach presented here, the generalized trace and cyclic properties arise in a purely natural way, without having to make any artificial interventions by hand. In Refs.[18],[41],[42] an attempt was made to get trace and cyclic properties, satisfied by an integral defined on κ-Minkowski spacetime. This attempt consisted in finding an appropriate integration meassure which would enable integral to have desired properties. However, described procedure appeared to have few stumbling blocks and did not solve the problem in a completely satisfactory way. In the approach carried out in this paper, the generalized trace and cyclic properties instead emerge quite naturally, simply by demanding that the classical Dirac operator representation (10) has to be hermitian, giving rise to hermitian realization (72). The star product corresponding to this realization then happens to have these nice properties, required for building any gauge theory.

It is obvious from relation (113) that the integral will exhibit standard trace property, provided that classical physical fields ψ and ϕ satisfy the conditions $\psi^{*\dagger} = \psi$ and $\phi^{*\dagger} = \phi$. In this case we would thus have

$$\int d^n x \, \psi^\dagger \star_h \phi = \int d^n x \, \phi \star_h \psi^\dagger. \tag{114}$$

In relation (97) we have calculated the corresponding integral for one definite order of plane wave factors. If the plane wave factors within this integral are reversed, the result would be

$$\int d^n x \, e^{iQx} \star_h e^{iS(P)x} = (2\pi)^n \, \delta^{(n)}(S(P) - S(Q)). \tag{115}$$

The right hand side of Eq.(115) can be calculated with the help of the known result for the Jacobian (107) to give

$$\delta^{(n)}(S(P) - S(Q)) \;=\; Z^{-(n-1)}(P)\,\delta^{(n)}(P - Q). \tag{116}$$

This would consequently lead to a generalized trace property at the level of plane waves,

$$\int d^n x \; e^{iS(P)x} \star_h e^{iQx} \;=\; Z^{n-1}(P) \int d^n x \; e^{iQx} \star_h e^{iS(P)x}, \tag{117}$$

where $Z^{n-1}(P)$ is a function of P, the same one that appears in (109). If the use is further made of Fourier expansions (50) and (53), and $Z^{n-1}(P)$ is transcribed into a form of differential operator (by simply setting $P_\mu = -i\partial_\mu$), then Eq.(117) would imply an alternative form of the identity embracing the generalized trace property,

$$\int d^n x \; \psi^\dagger \star_h \phi \;=\; \int d^n x \; (Z^{n-1}\phi) \star_h \psi^\dagger \;=\; \int d^n x \; \phi \star_h Z^{-(n-1)}\psi^\dagger, \tag{118}$$

where Z^{n-1} is now a differential operator, with Z^{-1} given in (14). The expression to the most right in relation (118) is a direct consequence of the property (32) of the antipode for translation generators.

As promised before, we turn to discussion regarding the issue of translation invariance for theories defined on κ-spaces. With this purpose, we recall that the star product (40), i.e. (48), with the hermitization procedure not being implemented in, is translationally invariant. This we inferred by following the arguments made in Refs.[15],[48] and by invoking that κ-Minkowski space is invariant [48] under translations, $\hat{x}_\mu \longrightarrow \hat{x}_\mu + \hat{y}_\mu$, provided that following conditions are satisfied:

$$[\hat{y}_\mu, \hat{y}_\nu] = i(a_\mu \hat{y}_\nu - a_\nu \hat{y}_\mu), \qquad [\hat{x}_\mu, \hat{y}_\nu] = 0. \tag{119}$$

Using given arguments we were able to show that the star product in nonhermitian realization (see Eq.(10)) is translationally invariant,

$$e^{iP(x+y)} \star e^{iQ(x+y)} \;=\; e^{i\mathcal{D}(P,Q)(x+y)}. \tag{120}$$

On the other side, in the case of the star product (90), i.e. (91) in hermitian realization (see Eq.(72)), translation $\hat{x}_\mu \longrightarrow \hat{x}_\mu + \hat{y}_\mu$ has led us to the conclusion

$$e^{iP(x+y)} \star_h e^{iQ(x+y)} \;=\; \left[\frac{\tilde{A}(\mathbf{K}^{-1}(P), Q)}{A(\mathbf{K}^{-1}(P))}\right]^2 e^{i\mathcal{D}(P,Q)(x+y)}, \tag{121}$$

showing that a redundant extra factor appears on the r.h.s. of Eq.(121), leading to a breaking of translational symmetry. This shows that the hermitization process on κ-Minkowski in some sence interferes with translation symmetry, at least in the setting given by the classical Dirac operator representation (11). Few comments are in order, regarding translation symmetry breaking:

1. It is evident from Eq.(121) that the condition for having translational invariance should be

$$A(\mathbf{K}^{-1}(P)) = 1. \tag{122}$$

   For the classical Dirac operator representation, i.e. that one characterized by (11), the condition (122) is obviously not satisfied. Thus, we have to conclude that classical Dirac operator realization, Eqs.(10),(11), is the one in which it is not possible to simultaneously have translation invariance property along with hermiticity. However, it does not mean that it is not possible to find realizations, even a whole family of realizations for which these two important requirements do not interfere. For example, in Ref.[31], the authors consider certain family of realizations which are hermitian and because satisfying the condition $[x_\mu, \Phi_{\nu\lambda}(\partial)] = 0$, have this $A(\mathbf{K}^{-1}(P))$ factor equal to 1 and consequently the star product constructed in that realizations will have the required translation invariance property, along with hermiticity. The analysis of these realizations, their Hopf algebraic descriptions and corresponding star products would be an interesting subject for future investigations.

2. In this paper we are considering a free scalar field theory whose action includes products of two fields. Because of the fundamental property (99) of the star product $\star_h$, established in this paper (removing a star product under integration sign), in the case of free scalar field theory we don't even in principle have a problem with translation invariance breaking.

3. In case of the interacting field theory, it seems that the breaking of translational symmetry is unavoidable. However, it is not clear at all what are the physical consequences of translational symmetry breaking. These consequences should be carefully investigated, especially the possible violation of energy-momentum conservation [73]. Even if this situation with translational symmetry breaking could cause violation of energy-momentum conservation, we point out that in the case of the star product (91), this violation would be so minute, that one could safely neglect it. This conclusion can be drawn from the form of the factor $A(\mathbf{K}^{-1}(P))$, whose lowest order corrections to its value are of the second order in deformation parameter a,

$$A(\mathbf{K}^{-1}(P)) = 1 + \mathcal{O}(a^2).$$

Thus, if $a = \frac{1}{\kappa}$ is taken to correspond to Planck length (where κ is the Planck mass), then a^2 contributions are really of a very small magnitude. If one would set up to analyse an interacting field theory on κ-space, constructed from the star product (91) in hermitian realization, then he would not encounter even this minute signal of energy-momentum conservation violation in case that he restricts his calculations to a first order corrections in deformation parameter a. Thus, when considering the interacting field theory on κ-space constructed in terms of star product (91), we do not expect a violation of energy-momentum conservation to show up within the first order in deformation parameter a.

As far as the issue of Lorentz invariance of the field theory on κ-Minkowski space is concerned, there is no definite answer yet in a response to this question. In most literature on the subject of κ-deformation, the deformation vector a_μ is treated as a vector that obeys standard rules for raising and lowering indices, realized through the application of the metric tensor, but anyway its components are treated as constant parameters in every Lorentz frame. We point out that such view inevitably leads to Lorentz violation. In reference [72] the authors also noticed this problem. Their actual momentum space for field theory was determined by the region in de Sitter space which is defined by the condition that is not Lorentz invariant. Following these conclusions they argue that it is to expect that any theory with such momentum space will suffer from Lorentz violation. Anyway, there has been an attempt made [74] in order to overcome this problem which, according to Ref.[74], was successfully solved by modifying the action of Lorentz generators. This modification, as claimed in [74], avoids the problem of Lorentz breaking.

However, we want to stress that in our approach the only possible way in which Lorentz symmetry can be restored is to treat deformation vector a_μ as a pure n-vector, which, besides obeying raising and lowering of indices by means of the metric tensor, indeed transforms as a n-vector under Lorentz transformations,

$$[M_{\mu\nu}, a_\lambda] = a_\mu \eta_{\nu\lambda} - a_\nu \eta_{\mu\lambda}. \tag{123}$$

Here we assume that $\hat{x}_\mu$ and a_ν commute among themselves, $[\hat{x}_\mu, a_\nu] = [a_\mu, a_\nu] = 0$. This would imply that NC coordinates also transform as a n-vector under Lorentz transformations, so that instead of relation (3), we would now have

$$[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \eta_{\nu\lambda} - \hat{x}_\nu \eta_{\mu\lambda}, \tag{124}$$

with all other algebraic relations between the generators $\hat{x}_\mu$, $M_{\mu\nu}$ and P_μ remaining unchanged [50]. If we are about to keep the representation (10) (or (72) in the case of hermitian realization) for NC coordinates intact, then the algebraic setting just described requires a modification of representation (8) for the Lorentz generators, so that a new form they acquire would look as

$$M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu + a_\mu \frac{\partial}{\partial a_\nu} - a_\nu \frac{\partial}{\partial a_\mu}. \tag{125}$$

The relation (124), comprising vector-like properties of NC coordinates, would have a far reaching consequence on the coalgebraic sector of the Hopf κ-Poincaré algebra. Particularly, it would affect a Lorentz part of the coalgebra by greatly simplifying it, giving rise to an undeformed coproduct for Lorentz generators,

$$\triangle M_{\mu\nu} \;=\; M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}, \tag{126}$$

replacing the relation (23). On the other hand, the coproduct for translation generators would remain unchanged, still given by the relation (22). In this case the symmetry properties of the field theory constructed in a described setting, would be enclosed by a Hopf algebra whose Lorentz symmetry is undeformed at both algebraic and coalgebraic level. Similar property is observed in Refs.[75],[76] in the context of Snyder spacetime. All deformations then would be encoded within the coproduct (22) for translation generators only, which in particular is coassociative. The corresponding star product (91) would consequently be associative, with homomorphism relating corresponding structures. Similar ideas concerning Lorentz covariance are presented in Ref.[77], where the authors consider Lorentz covariant κ-Minkowski spacetime. However, to find out whether Lorentz symmetry is really preserved or is actually broken below the Planck scale, we should wait for an experiment to get right and definite answer.

In this paper we have presented a construction of the star product on noncommutative κ-Minkowski spacetime in the setting provided by the classical Dirac operator representation. The construction alone relies on the very property of hermiticity and based on this property directly leads to a formulation of invariant integral having generalized trace property. This property is essential for building any gauge theory. To our knowledge, it is for the first time in the literature that invariant integral on κ-deformed space, having properties (99) and (113), has been constructed. However, obtained star product is not translationally invariant, showing internal incompatibility between properties of translation symmetry and hermiticity on κ-deformed spaces, at least for the classical Dirac operator representation.

Acknowledgment

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